PRACTICE FINAL (ADIREDJA) - SOLUTIONS

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1. (a) The fact that this limit does not exist shows that $|\sin(x)|$ is not **differentiable** at π. In fact, it *is* continuous at π as a composition of continuous functions!

   (b) The integral is 0 because the function is an odd function.

   (c) f is **minimized** at 1 ! $f'(x) = x^2 - 1$, so $f'(1) = 0$ and $f''(1) > 0$, so by the second derivative test, $f(1)$ is a minimum.

   (d) Yes you can! Take $\ln$s of both sides and write $\cos$ as $\frac{1}{\frac{1}{\cos(x)}}$, and use l'Hopital's rule.

   (e) Yes you can, by the extreme value theorem!

2. (a)

$$\int \frac{1-x}{\sqrt{1-x^2}}dx = \int \frac{1}{\sqrt{1-x^2}}dx - \frac{x}{\sqrt{1-x^2}}dx = \sin^{-1}(x) + \sqrt{1-x^2} + C$$

   where in the last step, we used the substitution $u = 1 - x^2$.

   (b) $\Delta x = \frac{2}{n}$, $x_i = a + i\Delta x = \frac{2i}{n}$

Date: Monday, May 9th, 2011.

$$\begin{aligned}
\int_0^2 2-x^2dx &= \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\Delta x\\
&= \lim_{n\to\infty}\sum_{i=1}^n \left(2-\left(\frac{2i}{n}\right)^2\right)\frac{2}{n}\\
&= \lim_{n\to\infty}\sum_{i=1}^n \frac{4}{n}-\sum_{i=1}^n\frac{8i^2}{n^3}\\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^n 1-\frac{8}{n^3}\sum_{i=1}^n i^2\\
&= \lim_{n\to\infty}\frac{4}{n}n-\frac{8}{n^3}\frac{n(n+1)(2n+1)}{6}\\
&= \lim_{n\to\infty} 4-\frac{8}{6}\frac{(n+1)(2n+1)}{n^2}\\
&=4-\frac{4}{3}(2)\\
&=4-\frac{8}{3}\\
&=\frac{4}{3}
\end{aligned}$$

(c)

$$\int_{-2}^2 f(x)dx=\int_{-2}^0 x+2dx+\int_0^2\sqrt{4-x^2}dx=\frac{2^2}{2}+\frac{1}{4}\pi 2^2=2+\pi$$

Where we used the fact that the first integral represents the area of a triangle with base 2 and height 2, and the second integral represents the area of a quarter circle of radius 2.

3. (a) (i)

$$\lim_{x\to 2}x^2-4x+7=4-8+7=3$$

(ii)

$$\lim_{x\to 2}x^2=4$$

(b) (i) Let $f(x)=x^2-4x+7$

Part I: Finding δ

1) $|f(x)-3|=\left|x^2-4x+7-3\right|=\left|x^2-4x+4\right|=|x-2|^2$
2) $|x-2|^2<\epsilon$ implies $|x-2|^2<\sqrt{\epsilon}$
3) Let $\boxed{\delta=\sqrt{\epsilon}}$

Part II: Showing your δ works

1) Let $\epsilon > 0$ be given. Let $\delta = \sqrt{\epsilon}$, and suppose $0 < |x-2| < \delta$. Then $|x-2| < \sqrt{\epsilon}$
2) Then $|f(x) - 3| = |x-2|^2 < (\sqrt{\epsilon})^2 = \epsilon$
3) Hence, if $0 < |x-2| < \delta$, then $|f(x) - 3| < \epsilon$

(ii) Let $f(x) = x^2$

Part I: Finding δ

1) $|f(x) - 4| = \left|x^2 - 4\right| = |x-2|\,|x+2|$
2) If $|x-2| < 1$, then $-1 < x - 2 < 1$, so $1 < x+2 < 3$, so $|x+2| < 3$
2) So $|f(x) - 4| = |x-2|\,|x+2| < 3\,|x-2| < \epsilon$ implies $|x-2| < \frac{\epsilon}{3}$
3) Let $\boxed{\delta = min\left\{1, \frac{\epsilon}{3}\right\}}$

Part II: Showing your δ works

1) Let $\epsilon > 0$ be given. Let $\delta = min\left\{1, \frac{\epsilon}{3}\right\}$, and suppose $0 < |x-2| < \delta$. Then $|x-2| < \frac{\epsilon}{3}$ and $|x+2| < 3$
2) Then $|f(x) - 4| = |x-2|\,|x+2| < 3\,|x-2| = 3\left(\frac{\epsilon}{3}\right) = \epsilon$
3) Hence, if $0 < |x-2| < \delta$, then $|f(x) - 4| < \epsilon$

4. (a)

$$\lim_{x\to 0} \frac{\tan^2(x)}{x^2} \overset{=}{H}= \lim_{x\to 0} \frac{\tan(x)\sec^2(x)}{x} = \lim_{x\to 0} \frac{\tan(x)}{x} \overset{=}{H}= \lim_{x\to 0} \sec^2(x) = 1$$

Where H means l'Hopital's rule. Also, in the second step, we used the fact that $\lim_{x\to 0} \sec^2(x) = 1$, so it doesn't affect our limit!

(b)

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{x}{x\sqrt{1+\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} = 1$$

Where we used the fact that $\sqrt{x^2} = |x| = x$ (since $x > 0$)

(c) Notice that if you let $f(x) = \int_x^{3x} t^2 \sin(t)dt$, then the limit is just $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$. So the answer is: $f'(x) = \boxed{3(3x)^2\sin(3x) - x^2\sin(x)}$.

You could also have used l'Hopital's rule, but be careful that you're differentiating with respect to h and not with respect to x here!!! (so $\int_x^{3x} t^2 \sin(t)dt$ is a **CONSTANT** in this case!)

5.
- Domain: $x \neq 0$
- Asymptotes: $y = 1$ (at $\pm\infty$, since $\lim_{x\to\pm\infty} e^{\frac{1}{x}} = 1$), $x = 0$ (since $\lim_{x\to 0^+} e^{\frac{1}{x}} = \infty$)
- $f'(x) = \frac{-1}{x^2}e^{\frac{1}{x}}$, no critical points, Decreasing on $(-\infty, 0)$ and on $(0, \infty)$
- No local extrema
- $f''(x) = \frac{2x+1}{x^4}e^{\frac{1}{x}}$, Concave down on $(-\infty, -\frac{1}{2})$, Concave up on $(-\frac{1}{2}, 0)$ and on $(0, \infty)$
- Inflection point: $(-\frac{1}{2}, e^{-2})$
- Graph: Check it with your calculator
- Range: $[0, 1) \cup (1, \infty)$ (look at your graph to convince yourself of this!)

6. (a)

1) Want to minimize $\sqrt{x^2 + \left(y - \frac{1}{2}\right)^2}$, same as minimizing $x^2 + \left(y - \frac{1}{2}\right)^2$, but $y = x^2 - 4$, so $\boxed{f(x) = x^2 + \left(x^2 - \frac{9}{2}\right)^2}$

2) Notice the symmetry in your picture! That's why we set our constraint to be $x > 0$

3) $f'(x) = 2x + 2\left(x^2 - \frac{9}{2}\right)(2x) = 2x\left(2x^2 - 8\right)$, $f'(x) = 0 \Leftrightarrow x = 0$ or $x = \pm 2$. But since $x > 0$, we only care about $\boxed{x = 2}$

4) By FDTAEV, $x = 2$ is an absolute minimum of f. and notice that $f(2) = 0$, so our answer is: $\boxed{(2, 0) \text{ and } (-2, 0)}$ (by symmetry)

(b) Look at your picture in (a), and notice the symmetry again! By symmetry, we only need to focus on the right hand side of the picture. The line connecting $(0, \frac{1}{2})$ and $(2, 0)$ has equation $y = -\frac{x}{4} + \frac{1}{2}$, so the area of the right hand side is:

$$A^+ = \int_0^2 \left(-\frac{x}{4} + \frac{1}{2}\right) - (x^2 - 4)dx = \int_0^2 -x^2 - \frac{x}{4} + \frac{9}{2}dx = \frac{35}{6}$$

So our answer is $\boxed{A = 2A^+ = \frac{35}{3}}$

7. Consider $f(t)$ and $g(t)$, the position functions of the runners. Define $\boxed{h(t) = f(t) - g(t)}$. Then $h(0) = f(0) - g(0) = 0$ (since the runners started at the same place), and $h(T) = f(T) - g(T) = 0$, where T is the ending time (we know $f(T) = g(T)$ because the race ended in a tie). But then $h(0) = h(T)$, so by Rolle's theorem, $\boxed{h'(c) = 0}$ for some c in $(0, T)$. But $h'(c) = f'(c) - g'(c)$, so $f'(c) - g'(c) = 0$, so $f'(c) = g'(c)$, but this says precisely that at some point in time (namely at c), the two runners had the same speed!